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Representations of $SU(2)$ in the plane

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Abstract

We present a family of unitary irreducible representations of $SU(2)$ realized in the plane, in terms of the Laguerre polynomials. These functions are similar to the Spherical Harmonics realized on the sphere. Relations with the space of square integrable functions defined on the plane, $L^2(\mathbb{R}^2)$, are analyzed. The realization of representations of Lie groups in spaces with intrinsic symmetry different from the one of the groups is discussed.

Keywords: Special functions, Laguerre polynomials, Lie algebras, Square-integrable functions

1 Introduction

The representations of a Lie algebra are usually considered as ancillary to the algebra and developed starting from the algebra, i.e. from the generators and their commutation relations. The universal enveloping algebra (UEA) is constructed and a complete set of commuting observables selected, choosing between the invariant operators of the algebra and of a chain of its subalgebras. The common eigenvectors of this complete set of operators are a basis of a vector space where the Lie algebra generators (and, in this way, all elements of the UEA) are realized as operators.

We propose here an alternative construction that allows to add to the representations obtained following the reported recipe, new ones not achievable following the historical approach:

a) we start from a concrete vector space of functions, where discrete labels and continuous variables are defined.

b) In this vector space we consider the recurrence relations that allow to connect functions with different values of the labels. As well known, these recurrence relations have been often connected with Lie algebras (see [1] and references therein).

c) These recurrence relations are not operators: successive applications of them require a change by hand of the values of the labels modified by previous applications. This means that the product is not defined on the recurrence relations.

d) We introduce for each label and for each continuous variable an operator that reads its value. Because recurrence relations have almost always a derivative contribution we include for each continuous variable also the derivative operator related to it . As discussed in [2, 3] the presence of operators with spectrum of different cardinality implies that, as considered for the first time in Lie algebras in [4], the space of the group representation is not a Hilbert space but a rigged Hilbert space [5].

e) Recurrence relations are rewritten in terms of rising and lowering operators built by means of the above defined operators.

f) These rising and lowering operators are often genuine generators of the Lie algebras considered by Miller [1] and the procedure gives simply the representations of the algebras in a well defined function space (see [6, 7, 8]).

g) However it can happen that the commutators, besides the values required by the algebra, have additional contributions. The essential point of this paper is that these additional contributions (as exhibited here) can be proportional to the null identity that defines the starting vector space. As this identity is zero on the whole representation, the Lie algebra is well defined and a new representation in a space of functions has been found.

We do not discuss here the general approach, but we limit ourselves to a simple example where all aspects are well understandable. We start thus from the associated Laguerre functions and, following the proposed construction, we realize the algebra $su(2)$ defined by the appropriate rising and lowering operators. Associated Laguerre functions

support in reality a larger algebra [9] but we prefer to consider here only the sub-algebra $su(2)$. The reasons for this choice are twofold: first in this way the technicalities are reduced at the minimum and second it has been very nice for us to discover that not all representations of a so elementary group like $SU(2)$ where known.

Thus, in the plane $\mathbb{R}^2$, defined by $\{y \in \mathbb{R}|0 \leq y \leq \infty\} \otimes \{\phi \in \mathbb{R}| -\pi \leq \phi \leq \pi\}$, a representation of $SU(2)$ is obtained in terms of operators that do not close, as operators, the algebra $su(2)$. However when these operators are applied to the appropriate space of representation (the square integrable functions in the plane $L^2(\mathbb{R}^2)$), where one operator of the UEA is identically zero the representation of $SU(2)$ is obtained.

2 Associated Laguerre polynomials

We start presenting the pertinent properties of the associated Laguerre polynomials $L_n^{(\alpha)}(y)$ [10]. They depend from a continuous variable y, $\{y \in \mathbb{R}|0 \leq y \leq \infty\}$, and from two real labels $n \in \mathbb{N}$ $(n = 0, 1, 2, \cdots)$ and α (usually assumed as a fix parameter), continuous and > -1. They reduce to the Laguerre polynomials for $\alpha = 0$ and are defined by the second order differential equation

$$\left[y\frac{d^2}{dy^2} + (1+\alpha-y)\frac{d}{dy} + n\right] L_n^{(\alpha)}(y) = 0\ . \tag{1}$$

From the many recurrence relations that can be found in literature [10, 11], we consider the following ones, all first order differential recurrence relations:

$$\begin{aligned}
&\left[y\frac{d}{dy} + (n+1+a-y)\right] L_n^{(\alpha)}(y) = (n+1)L_{n+1}^{(\alpha)}(y) \\
&\left[-y\frac{d}{dy} + n\right] L_n^{(\alpha)}(y) = (n+\alpha)L_{n-1}^{(\alpha)}(y) \\
&\left[-\frac{d}{dy} + 1\right] L_n^{(\alpha)}(y) = L_n^{(\alpha+1)}(y) \\
&\left[y\frac{d}{dy} + \alpha\right] L_n^{(\alpha)}(y) = (n+\alpha)L_n^{(\alpha-1)}(y)\,.
\end{aligned} \tag{2}$$

Starting from $L_n^{(\alpha)}(y)$, by means of repeated applications of eqs. (2), where after each application the values of n and α are modified by hand in the subsequent ones, $L_{n+k}^{(\alpha+h)}(y)$ –with h and k arbitrary integers– can be obtained, through a differential relation of higher order. However, by means of eq. (1), every differential relation of order two or more can be rewritten as differential relation of order one. In other words, the transformation $\mathcal{O}_k^h$, such that,

$$\mathcal{O}_k^h\ L_n^{(\alpha)}(y) = L_{n+k}^{(\alpha+h)}(y),$$

can be always written as first order differential relation. In particular we can obtain

$$\begin{aligned} &\left[\frac{d}{dy}+\frac{n}{\alpha+1}\right] L_n^{(\alpha)}(y) \;=\; -\frac{\alpha}{\alpha+1}\, L_{n-1}^{(\alpha+2)}(y), \\ &\left[y(\alpha-1)\frac{d}{dy}-y\left(n+3\,\frac{\alpha}{2}\right)+\alpha(\alpha-1)\right] L_n^{(\alpha)}(y)=(j+\alpha)(\alpha+1)\, L_{n+1}^{(\alpha-2)}(y), \end{aligned} \tag{3}$$

that are the recurrence relations we employ in this paper.

The associated Laguerre polynomials $L_n^{(\alpha)}(y)$ are –for $\alpha > -1$ and fix– orthogonal in n with respect the weight measure $d\mu(y) = y^\alpha\, e^{-y}\, dy$ [10, 12]:

$$\begin{aligned} &\int_0^\infty dy\; y^\alpha\; e^{-y}\, L_n^{(\alpha)}(y)\; L_{n'}^{(\alpha)}(y) \;=\; \frac{\Gamma(n+\alpha+1)}{n!}\,\delta_{nn'}\;, \\ &\sum_{n=0}^{\infty} y^\alpha\, e^{-y}\, L_n^{(\alpha)}(y) L_n^{(\alpha)}(y') = \delta(y-y')\,. \end{aligned} \tag{4}$$

In reality, as discussed in details in [10], the parameter α can be extended to arbitrary complex values and, in particular, for α integer and such that $0 \leq |\alpha| \leq n$, this generalization allows to find the relation

$$L_n^{(-\alpha)}(y) = (-y)^\alpha \frac{(n-\alpha)!}{n!}\, L_{n-\alpha}^{(\alpha)}(y)\,. \tag{5}$$

In this paper, we assume consistently

$$n \in \mathbb{N}\,, \qquad \alpha \in \mathbb{Z}\,, \qquad n-\alpha \in \mathbb{N}\,, \tag{6}$$

and we consider α as a label, like n, and not a parameter fixed at the beginning.

Following the approach of [6, 7], we introduce now a set of alternative variables and include the weight measure inside the functions, in such a way to obtain the bases we are used in quantum mechanics. We define indeed

$$j := n + \frac{\alpha}{2}\,, \qquad m := -\frac{\alpha}{2}\,,$$

that, from eqs. (6), are such that

$$j \in \mathbb{N}/2\,, \qquad j-m \in \mathbb{N}\,, \qquad |m| \leq j$$

i.e. look like the parameters j and m we are used in $SU(2)$.

Now we write

$$\mathcal{L}_j^m(y) := \sqrt{\frac{(j+m)!}{(j-m)!}}\; y^{-m}\, e^{-y/2}\, L_{j+m}^{(-2m)}(y)$$

so that, from eq. (5), $\mathcal{L}_j^m(y)$ are symmetric in the exchange $m \leftrightarrow -m$

$$\mathcal{L}_j^m(x) = \mathcal{L}_j^{-m}(x). \tag{7}$$

Eq.(1) can be rewritten on the $\mathcal{L}_j^m(y)$ as

$$\left[y \frac{d^2}{dy^2} + \frac{d}{dy} - \frac{m^2}{y} - \frac{y}{4} + j + \frac{1}{2} \right] \mathcal{L}_j^m(y) = 0. \tag{8}$$

From eqs. (4), the $\{\mathcal{L}_j^m(y)\}$ verify, for m fix, the orthonormality relation

$$\int_0^\infty \mathcal{L}_j^m(y)\ \mathcal{L}_{j'}^m(y)\ dx \ = \delta_{jj'}$$

$$\sum_{j=|m|}^\infty \mathcal{L}_j^m(y)\ \mathcal{L}_j^m(y') \ = \delta(y - y')$$

and are thus, for any fix value of m, an orthonormal basis of the space of square integrable functions on the half-line $L^2(\mathbb{R}^+)$ [12].

Note that, in the algebraic description of spherical harmonics, the functions $T_j^m(x)$, related to the associated Legendre functions $P_l^m(x)$,

$$T_j^m(x) := \sqrt{\frac{(j-m)!}{(j+m)!}}\, P_j^m(x),$$

have been introduced [7], such to satisfy a relation similar to (7), i.e.

$$T_j^m(x) = (-1)^m\, T_j^{-m}(x)$$

and that, like the $\mathcal{L}_j^m(y)$ on the half-line, are orthogonal –for fix m– in the interval $(-1,+1)$ and a basis for $L^2(x)$ for $\{x \in \mathbb{R} | -1 \leq x \leq +1\}$.

3 Construction of the representations

Following now Ref. [6], we define four operators Y, D_y, J and M such that

$$\begin{aligned} Y\ \mathcal{L}_j^m(y) \ &= \ y\ \mathcal{L}_j^m(y), & D_y\ \mathcal{L}_j^m(y) \ &= \ \mathcal{L}_j^m(y)'\ , \\ J\ \mathcal{L}_j^m(y) \ &= \ j\ \mathcal{L}_j^m(y), & M\ \mathcal{L}_j^m(y) \ &= \ m\ \mathcal{L}_j^m(y). \end{aligned}$$

and rewrite eq.(8) in operatorial form as

$$\left[Y\, D_y^2 + D_y - \frac{1}{Y} M^2 - \frac{Y}{4} + J + \frac{1}{2} \right] \mathcal{L}_j^m(y) = 0\,.$$

The identity $E \equiv 0$ with

$$E \; := \; Y\,D_y^2 + D_y - \frac{1}{Y}M^2 - \frac{Y}{4} + J + \frac{1}{2} \tag{9}$$

defines thus the space of square integrable functions on the half-line $L^2(\mathbb{R}^+)$.

The relations (3) can now be written on the $\mathcal{L}_j^m(y)$:

$$\begin{aligned} K_+\,\mathcal{L}_j^m(y) &= \sqrt{(j-m)(j+m+1)}\;\;\mathcal{L}_j^{m+1}(y)\,, \\ K_-\,\mathcal{L}_j^m(y) &= \sqrt{(j+m)(j-m+1)}\;\;\mathcal{L}_j^{m-1}(y)\,, \end{aligned} \tag{10}$$

where

$$\begin{aligned} K_+ &= -2D_y\left(M+\frac{1}{2}\right)+\frac{2}{Y}M\left(M+\frac{1}{2}\right)-\left(J+\frac{1}{2}\right)\,, \\ K_- &= 2D_y\left(M-\frac{1}{2}\right)+\frac{2}{Y}M\left(M-\frac{1}{2}\right)-\left(J+\frac{1}{2}\right)\,. \end{aligned} \tag{11}$$

As, from eqs. (10), we have

$$[K_+, K_-]\;\mathcal{L}_j^m(y) \;=\; 2\,m\,\mathcal{L}_j^m(y),$$

we assume $K_3 \;:=\; M$, so that

$$K_3\;\mathcal{L}_j^m(y) = \; m\;\mathcal{L}_j^m(y)\,,$$

and we write

$$\begin{aligned} [K_+, K_-]\;\mathcal{L}_j^m(y) &= 2\,K_3\;\mathcal{L}_j^m(y), \\ [K_3, K_\pm]\,\mathcal{L}_j^m(y) &= \pm\,K_\pm\;\mathcal{L}_j^m(y) \end{aligned}$$

exhibiting that, for fix j, under the action of $\{K_+, K_3, K_-\}$, the space $\{\mathcal{L}_j^m(y)\}$ looks like the unitary irreducible representation of $su(2)$ of dimension $2j+1$.

The structure can be analyzed more in detail: $\{K_+, K_3, K_-\}$ act on $\mathcal{L}_j^m(y)$ as operators with $\Delta(j) = 0$ and $\Delta(m) = -1, 0, +1$, so that also the operators D_y and $1/Y$ must act in analogous way. The hermiticity relation $K_\mp = K_\pm^\dagger$ implies

$$[J, \bullet]\;\mathcal{L}_j^m(y) \;=\; 0\,, \qquad\qquad [M, D_y]\;\mathcal{L}_j^m(y) \;= -D_y\,\mathcal{L}_j^m(y)\,,$$

$$\left[M, \frac{1}{Y}\right]\;\mathcal{L}_j^m(y) \;=\; -\;\frac{1}{2Y}\;\mathcal{L}_j^m(y)\;, \qquad\qquad \left[D_y, \frac{1}{Y}\right]\;\mathcal{L}_j^m(y) \;= -\frac{1}{Y^2}\;\mathcal{L}_j^m(y)\,.$$

However, as reported in eq. (7), we have the problem that $\mathcal{L}_j^m(y) = \mathcal{L}_j^{-m}(y)$ i.e. the vector space is incorrectly defined (as it happens also with the $P_j^m(x)$), because the

eigenfunctions related to $\pm m$ coincide. We remove the degeneracy, like it has been done in spherical harmonics, considering the new objects

$$\mathcal{Z}_j^m(y,\phi) := e^{im\phi}\,\mathcal{L}_j^m(y),$$

with $\{\phi \in \mathbb{R}| -\pi \leq \phi \leq \pi\}$. The functions $\mathcal{Z}_j^m(y,\phi)$ are thus the analogous on the plane of the spherical harmonics $Y_{lm}(\theta,\phi)$ on the sphere. Normalization and orthogonality of the $\mathcal{Z}_j^m(y,\phi)$ are similar to the ones of $Y_j^m(\theta,\phi)$

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \int_0^\infty \; dy \;\; \mathcal{Z}_j^m(y,\phi)^* \; \mathcal{Z}_{j'}^{m'}(y,\phi) \; &= \; \delta_{j,j'} \;\; \delta_{m,m'} \; , \\
\sum_{j,m} \;\; \mathcal{Z}_j^m(y,\phi)^* \; \mathcal{Z}_j^m(y',\phi') \; &= \; \delta(y-y')\,\delta(\phi-\phi')\,.
\end{aligned} \tag{12}$$

This means that the set $\{\mathcal{Z}_j^m(y,\phi)\}$ is a basis of the Hilbert space of $L^2(\mathbb{R}^2)$, like $\{Y_j^m(\Omega)\}$ is a basis of the Hilbert space of square integrable functions defined on the sphere $L^2(\mathbb{S}^2)$.

In reality, because of the different cardinality of y and j, we are in a rigged Hilbert space [2, 3] and

$$\mathcal{Z}_j^m(y,\phi) = \langle j,m|y,\phi\rangle$$

are the transformation matrices from the irreducible representation states $\{|j,m\rangle\}$ to the localized states in the plane $\{|y,\phi\rangle\}$, like

$$Y_j^m(\theta,\phi) = \langle j,m|\theta,\phi\rangle$$

are the corresponding ones to the localized states $\{|\theta,\phi\rangle\}$ in the sphere [13]. Indeed

$$|j,m\rangle \; = \; \int |y,\phi\rangle \; \mathcal{Z}_j^m(y,\phi)^* \; dy \; d\phi \; ,$$

$$|j,m\rangle \; = \; \int |\theta,\phi\rangle \; Y_j^m(\theta,\phi)^* \; d\Omega \; .$$

We continue with the analogy and, from the operators $\{K_+, K_3, K_-\}$ of eq. (11), we define

$$J_\pm \; := \; e^{\pm i\phi}\,K_\pm, \qquad\quad J_3 \; := \; K_3, \tag{13}$$

with act on the $\mathcal{Z}_j^m(y,\phi)$ as

$$\begin{aligned}
J_+ \;\; \mathcal{Z}_j^m(y,\phi) \;\; &= \;\; \sqrt{(j-m)(j+m+1)} \;\; \mathcal{Z}_j^{m+1}(y,\phi), \\
J_- \;\; \mathcal{Z}_j^m(y,\phi) \;\; &= \;\; \sqrt{(j+m)(j-m+1)} \;\; \mathcal{Z}_j^{m-1}(y,\phi), \\
J_3 \;\; \mathcal{Z}_j^m(y,\phi) \;\; &= \;\; m\;\mathcal{Z}_j^m(y,\phi)\,.
\end{aligned} \tag{14}$$

All problems appear to be solved as the functions $\mathcal{Z}_j^m(y,\phi)$ with j fix and $|m| \leq j$, are orthonormal and determine the representation of dimension $2j+1$ of $su(2)$ as it happens for the $Y_j^m(\theta,\phi)$.

However there is a fundamental difference between the operators $\{J_+, J_-, J_3\}$ that act on the sphere $\mathbb{S}^2$ [13] that are true generators of $su(2)$ and the operators $\{J_+, J_-, J_3\}$ of eq. (13), defined in $\mathbb{R}^2$, that do not close a Lie algebra.

Indeed, when we calculate the commutators and the $su(2)$ invariant of the operators defined in the eqs. (13), we obtain

$$\begin{aligned}
[J_+, J_-] \;&=\; 2\, J_3 + \frac{1}{Y}\, J_3\, E \;, \\
J_3^2 + \frac{1}{2}\, \{J_+, J_-\} \;&=\; J(J+1) \;\; - \frac{1}{Y}\, J_3^2\; E \;,
\end{aligned}$$

where E is given by eq. (9), that are not the equations of the $su(2)$ algebra. Only when $E \equiv 0\,,$ i.e. only in the representation defined on $L^2(\mathbb{R}^2)$, the $su(2)$ structure is recovered.

4 Conclusion

Summarizing the procedure, we start from the recurrence relations of eq. (3) and we obtain the operators $\{J_+, J_-, J_3\}$ of eq. (13) and their ordered polynomials (closure included). The algebraic structure is complicated and it has nothing to do with $su(2)$. On the contrary, as the representation of these operators on the vector space $L^2(\mathbb{R}^2)$ is characterized by the eigenvalue zero of the operator E, the set of all $\mathcal{Z}_j^m(y,\phi)$ is isomorphic to the regular representation of $su(2)$, $\{|j,m\rangle\}$. The group $SU(2)$ is obtained, at this point, by simple exponentiation.

We are used in Lie algebras to representations that save the symmetry of the algebra and to algebras that have the same symmetry of the space where the representation is defined. This is exactly what happens with the spherical harmonics, that are solution of Laplace equation and, thus, have the same intrinsic symmetry of the group $SU(2)$ of which they are representation bases. However, the situation is different when we represent $SU(2)$ in the plane $\mathbb{R}^2$ that saves only the subgroup $SO(2)$ of $SU(2)$. To reproduce the spherical symmetry, $SO(2)$ must indeed to be implemented with the additional condition $E \equiv 0$. Indeed the operators $\{J_+, J_-, J_3\}$ of eq. (13) are defined for any value of E, but they generate $su(2)$ only under the assumption $E \equiv 0$.

The connection between the algebraic structure and the regular representation is shown thus to be not one-to-one, as the regular representation of a Lie algebra has been here related not only to the Lie algebra but also to a set of operators that do not close a Lie algebra but reduce to a Lie algebra only when the operator E is identically zero.

This paper opens a new line to develop representations of Lie groups in space that are not symmetric under the full group, but only under one of its subgroups.

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